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Geometrized Dynamics of Multidimensional Cosmological Models

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In the present work we reduce the dynamics of multidimensional cosmological models to the geodesics on a pseudo-Riemannian space. The significance of Killing vectors and tensors for the integrability problem is discussed. We also investigate geometric properties of the geodesics representing the evolution of cosmological models.

1. INTRODUCTION

Synge (1926) proposed the idea of geometry of dynamics in a different sense than the usual symplectic geometry. The symplectic structure concerns a phase space, whereas in Synge's approach one is interested in the geometry of certain regions of configuration space. This geometry reflects the intrinsic properties of dynamical equations themselves. In the present work we develop this idea as far as the dynamics is concerned of simple multidimensional cosmological models. This approach pays off with the possibility to analyze effectively integrability of the system considered. As will be shown, integrability of a Hamiltonian system (resulting from a "certain algebraic degeneracy") demonstrates itself in symmetries of a certain space of geodesics to which the system can be reduced. Killing vectors of an isometry group of this space determine *first integrals* linear in momenta, whereas Killing tensors lead to first integrals of higher orders in momenta.

We shall show that a class of multidimensional world models, with cosmological constant and with the topology of the total space being the product of maximally symmetric internal and physical spaces, is integrable. In our approach the integrability of the system is equivalent to the integrability of the geodesic equation on the reduced space. We shall

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demonstrate that the class of classical cosmological models with a scalar field (minimally coupled) is also integrable.

The considered class of dynamical systems (corresponding to the above cosmological models) can be described with the help of a generalized Liouville surface in pseudo-Euclidean spaces. The Liouville systems constitute an important class of integrable Hamiltonian system. Since Hamiltonian systems of general relativity are distinguished by having indefinite kinetic energy forms, there is a necessity to generalize the classical Liouville systems to the relativistic case; we shall call them *nonclassical Liouville systems*. The systems considered in the present work belong to this class. Such systems, in close parallel to their classical counterparts, admit first integrals of the second order in momenta. We show that this is a consequence of the fact that the Liouville systems admit Killing tensors.

In Section 2 we show how to reduce the dynamics to the geometry of geodesics on a (pseudo)Riemannian space. In Section 3 we demonstrate that the multidimensional cosmology can be reduced to the classical cosmology with scalar fields. The significance of Killing vectors and Killing tensors for finding additional first integrals (besides the energy integral) is discussed in Section 4. In Section 5 we investigate geometric properties of the geodesics on nonclassical Liouville spaces and show the integrability of corresponding systems. Some conclusions in Section 6 complete our work.

2. THE PRINCIPLE OF MAUPERTUIS–JACOBI–LAGRANGE AS A METHOD TO GEOMETRIZE DYNAMICS

In the following we shall consider dynamical systems with the Lagrange function of the form

$$L(q, \dot{q}) = \frac{1}{2} a_{ij}(q)\dot{q}^i\dot{q}^j - V(q) \tag{1}$$

$$\dot{q}_i = \frac{dq_i}{dt}(t) \tag{2}$$

where

$$B(\xi, \xi) = \frac{1}{2} a_{ij}\xi^i\xi^j \tag{3}$$

is a positive-definite function (in general we assume that metric is indefinite). Since the momenta are given by

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = a_{ij}(q)\dot{q}^j \tag{4}$$

then, if we define matrix a^{ij} such that

$$a^{ij}(q)a_{jr}(q) = \delta^i_r \tag{5}$$

for every q, one has

$$\dot{q}^j = a^{jk}(q)p_k \tag{6}$$

Therefore

$$L(p, q) = \frac{1}{2} a^{ij}(q)p_i p_j - V(q) \tag{7}$$

In such a case the Hamiltonian is

$$H(p, q) = p_i \dot{q}^i - L(p, q) = \frac{1}{2} a^{ij}(q)p_i p_j + V(q) \tag{8}$$

and the Hamiltonian equations are of the form

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}, \qquad \dot{q}^i = \frac{\partial H}{\partial p_i} \tag{9}$$

that is,

$$\dot{p}_i = -\frac{1}{2}\left[\frac{\partial}{\partial q^i} a^{rs}(q)\right] p_r p_s - \frac{\partial}{\partial q^i} V(q) \tag{10}$$

$$\dot{q}^i = a^{ij}(q)p_j \tag{11}$$

Since $p_i = a_{ik}(q)\dot{q}^k$, one has

$$\begin{aligned} \dot{p}_i = \frac{d}{dt}[a_{ik}(q)\dot{q}^k] &= a_{ik}\ddot{q}^k + \dot{q}^s\left(\frac{\partial a_{ik}}{\partial q^s}\right)\dot{q}^k \\ &= -\frac{1}{2}\left[\frac{\partial a^{rs}}{\partial q^i}\right] a_{rj} a_{ls} \dot{q}^j \dot{q}^l - \frac{\partial V}{\partial q^i} \end{aligned} \tag{12}$$

and therefore

$$\begin{aligned} \ddot{q}^m &= -a^{mi}\left(\frac{\partial a_{ik}}{\partial q^s}\right)\dot{q}^s\dot{q}^k - \frac{1}{2} a^{mi}\left(\frac{\partial a^{rs}}{\partial q^i}\right) a_{rj} a_{sl} \dot{q}^j \dot{q}^l - \frac{\partial V}{\partial \dot{q}^i} a^{mi} \\ &= -\frac{1}{2} a^{mi}\left\{\frac{\partial a_{ik}}{\partial q^s} + \frac{\partial a_{is}}{\partial q^k} - \frac{\partial a_{sk}}{\partial q^i}\right\}\dot{q}^s\dot{q}^k - \frac{\partial V}{\partial q^i} a^{mi} \end{aligned} \tag{13}$$

that is,

$$\ddot{q}^m + \hat{\Gamma}^m_{sk}(q)\dot{q}^s\dot{q}^k = -\frac{\partial V}{\partial q^i} a^{im} \tag{14}$$

where

$$\hat{\Gamma}^m_{sk} \equiv \frac{1}{2} a^{mi}\left\{\frac{\partial a_{ik}}{\partial q^s} + \frac{\partial a_{is}}{\partial q^k} - \frac{\partial a_{sk}}{\partial q^i}\right\}$$

As we can see, the geodesic equation (14) is a consequence of the Hamilton equation.

It can be easily checked that along the motion trajectories

$$\frac{\partial H}{\partial t} = 0 \tag{15}$$

and since $H(p, q) = h = \text{const}$, one has

$$H(p, q) = \frac{1}{2} a^{ij}(q) p_i p_j + V(q) = h = \text{const} \tag{16}$$

$$L(p, q) = -H(p, q) + p_i \dot{q}^i \tag{17}$$

Therefore along a trajectory,

$$L(p, q) = p_i \dot{q}^i - h \tag{18}$$

To find the motion equations, it is enough to extremize the simplified action

$$I = \int_{t_a}^{t_b} p_i \dot{q}^i \, dt \tag{19}$$

with the Hamiltonian constraint:

$$\frac{1}{2} a_{ij} \dot{q}^i \dot{q}^j + V(q) = h = \text{const}$$

and we obtain

$$I = \int_{t_a}^{t_b} p_i \dot{q}^i \, dt = \int_{t_a}^{t_b} a_{ij} \dot{q}^i \dot{q}^j \, dt \tag{20}$$

Now, we should, in the simplest possible manner, take into account constraint. We have

$$a_{ij} \dot{q}^i \dot{q}^j = 2[h - V(q)] = 2W \tag{21}$$

and with the new parametrization $t = t(\lambda)$

$$a_{ij}\{q[t(\lambda)]\} \frac{dq^i}{d\lambda} \frac{dq^j}{d\lambda} \left(\frac{d\lambda}{dt}\right)^2 = 2W\{q[t(\lambda)]\} \tag{22}$$

We assume that

$$a_{ij}\{q[t(\lambda)]\} \frac{dq^i}{d\lambda} \frac{dq^j}{d\lambda} = 1 \tag{23}$$

(which is also the definition of λ), which gives

$$dt = \frac{d\lambda}{(2W\{q[t(\lambda)]\})^{1/2}} \tag{24}$$

The choice of the parameter λ guarantees that a "particle" is on *the level surface* $H(p, q) = h$. This leads to

$$I = \int_{t_a}^{t_b} a_{ij}[q(t)]\dot{q}^i(t)\dot{q}^j(t)\,dt = 2\int_{t_a}^{t_b} W[q(t)]\,dt$$
$$= 2\int_{\lambda_a}^{\lambda_b} W[q(t)]\frac{dt}{d\lambda}\,d\lambda = \int_{\lambda_a}^{\lambda_b} (2W[q(\lambda)])^{1/2}\,d\lambda \tag{25}$$

where $Wq[t(\lambda)] \equiv W[q(\lambda)]$.

By using the definition of the parameter λ we obtain the action I, which is independent of the parametrization of "particle" trajectory C,

$$I = \int_C \{2[h - V(q)]a_{ij}\,dq^i\,dq^j\}^{1/2} \tag{26}$$

In this way the problem of motion is reduced to the extremization of the arc length of the curve with the help of the metric

$$g_{ij} = 2[h - V(q)]a_{ij}(q) \tag{27}$$

Let us note that in the case of indefinite metrics we should take modulus in the above formula, i.e.

$$g^{ij} = [2|h - V(q)]a_{ij}(q)$$

To perform the extremization, we introduce, as usual, a family of curves, "numbered" with a parameter u, having the same initial and endpoint. The condition for an extremum

$$\left.\frac{\partial I[u]}{\partial u}\right|_{u=0} = 0 \quad \text{and} \quad q'(\lambda) = \frac{dq^i}{d\lambda}(\lambda) \tag{28}$$

reads

$$\int_{\lambda_a}^{\lambda_b} \frac{\partial}{\partial u}[2\{h - V[q_u(\lambda)]a_{ij}[q_u(\lambda)]\}q_u'^i(\lambda)q_u'^j(\lambda)]^{1/2}\,d\lambda$$
$$= \int_{\lambda_a}^{\lambda_b} \frac{1}{2}\{g_{ij}[q_u(\lambda)]q_u'^i(\lambda)q_u'^j(\lambda)\}^{-1/2}$$
$$\times \frac{\partial}{\partial u}\{g_{ij}[q_u(\lambda)]q_u'^i(\lambda)q_u'^j(\lambda)\}\,d\lambda = 0 \tag{29}$$

Now, let us compute $(\partial/\partial u)\{g_{ij}[q_u(\lambda)]q_u'^i(\lambda)q_u'^j(\lambda)\}$. After elementary manipulations we have

$$\frac{\partial}{\partial u}\{g_{ij}[q_u(\lambda)]q_u'^i(\lambda)q_u'^j(\lambda)\}$$
$$= -2\left[g_{ij}q_u''^i(\lambda)\frac{\partial q^j_u}{\partial u} + \left(\frac{\partial g_{rj}}{\partial q^i_u}\right)q_u'^i q_u'^r\frac{\partial q^j_u}{\partial u}\right.$$
$$\left. - \frac{1}{2}\left(\frac{\partial g_{rs}}{\partial q^j_u}\right)q_u'^r q_u'^s\frac{\partial q^j_u}{\partial u}\right] + 2\frac{\partial}{\partial\lambda}\left[g_{ij}q_u'^i\frac{\partial q^j_u}{\partial u}\right]$$
$$= -2g_{ij}\left[q_u''^i + \frac{1}{2}g^{ip}\left(\frac{\partial g_{rp}}{\partial q^s_u} + \frac{\partial g_{sp}}{\partial q^r_u} - \frac{\partial g_{sr}}{\partial q^p_u}\right)q_u'^s q_u'^r\right]\frac{\partial q^j_u}{\partial u}$$
$$+ 2\frac{\partial}{\partial\lambda}\left[g_{ij}q_u'^i\frac{\partial q^j_u}{\partial u}\right] \tag{30}$$

Hence, if we define

$$\Gamma^i_{sr} \equiv \frac{1}{2} g^{ip}\left(\frac{\partial g_{rp}}{\partial q^s_u} + \frac{\partial g_{sp}}{\partial q^r_u} - \frac{\partial g_{sr}}{\partial q^p_u}\right) \tag{31}$$

we finally obtain

$$\begin{aligned}\frac{\partial I}{\partial u}&[u=0]\\ &= \int_{\lambda_a}^{\lambda_b} \{g_{ij}[q(\lambda)]q'^i q'^j\}^{-1/2}\left\{-g_{ij}[q''^i + \Gamma^i_{sr} q'^s q'^r]\frac{\partial q^j}{\partial u}(u=0)\right\} d\lambda\\ &\quad + \int_{\lambda_a}^{\lambda_b} \frac{(\partial/\partial\lambda)[g_{ij}q'^i(\partial q^j/\partial u)(u=0)]}{\{g_{ij}[q(\lambda)]q'^i q'^j\}^{1/2}} d\lambda\end{aligned} \tag{32}$$

Let us now choose a new parameter s such that

$$g_{ij}\frac{dq^i}{ds}\frac{dq^j}{ds} = 1, \qquad (\text{sign}(h-V) \text{ in general}) \tag{33}$$

The relationship between the new parameter and the old one is

$$a_{ij}\dot{q}^i[s(t)]\dot{q}^j[s(t)]\left(\frac{dt}{ds}\right)^2 = 1 \qquad (\text{sign}(h-V) \text{ in general}) \tag{34}$$

Hence

$$\frac{ds}{dt} = 2W; \qquad W = |h-V| \text{ in general} \tag{35}$$

In the new parametrization "s" we have

$$\begin{aligned}\frac{\partial I}{\partial u}[u=0] = \int_{s_a}^{s_b}&\left\{-g_{ij}[q''^i + \Gamma^i_{sr}q'^s q'^r]\frac{\partial q^j}{\partial u}(u=0)\right\} ds\\ &+ \int_{s_a}^{s_b} \frac{\partial}{\partial s}\left[g_{ij}q'^i\frac{\partial q^j}{\partial u}(u=0)\right] ds = 0\end{aligned} \tag{36}$$

Since, however, for $s = s_a$ and $s = s_b$, $(\partial q^i/\partial u)$ $(s = s_a$ or $s = s_b) = 0$, one has

$$\frac{\partial I}{\partial u}[u=0] = -\int_{s_a}^{s_b}\left\{g_{ij}[q''^i + \Gamma^i_{sr}q'^s q'^r]\frac{\partial q^j}{\partial u}(u=0)\right\} ds = 0 \tag{37}$$

and consequently (because the above holds for any variation $(\partial q^i/\partial u)$

$$\frac{d^2q^i}{ds^2} + \Gamma^i_{lm}[q(s)]\frac{dq^l}{ds}\frac{dq^m}{ds} = 0 \tag{38}$$

After returning to the previous time parameter t in the motion equations, we obtain

$$\ddot{q}^i + \Gamma^i_{km}\dot{q}^k\dot{q}^m = \frac{1}{W}\dot{q}^k(\partial_k W)\dot{q}^i \tag{39}$$

where

$$\frac{d}{ds} = -\frac{1}{4W^3}\frac{\partial W}{\partial q^i}\dot{q}^i\frac{d}{dt} \tag{40}$$

$$\frac{d^2}{ds^2} = -\frac{1}{4W^3}(\partial_i W)\dot{q}^i\frac{d}{dt} + \frac{1}{4W^2}\frac{d^2}{dt^2} \tag{41}$$

Let us now summarize our results obtained so far:

1. The dynamics of a system having the Lagrange function $L = \frac{1}{2}a_{ij}\dot{q}^i\dot{q}^j - V(q)$ with the fixed energy h has been reduced to determining the geodesics

$$\frac{d^2q^i}{ds^2} + \Gamma^i_{jk}\frac{dq^j}{ds}\frac{dq^k}{ds} = 0 \tag{42}$$

$$\frac{ds}{dt} = 2W\{q[s(t)]\}, \qquad W = |h - V(q)| \text{ in general.}$$

where

$$\Gamma^i_{lm}(q) = -\frac{1}{2W}[\partial_l V\delta^i_m + \partial_m V\delta^i_l - \partial_j V a^{ji} a_{lm}] + \frac{1}{2}a^{ij}[\partial_j a_{lm} + \partial_m a_{jl} - \partial_j a_{lm}] \tag{43}$$

If $a_{ij} = \delta_{ij}$, then $a^{ij} = \delta^{ij}$ and

$$\Gamma^i_{lm}(q) = -\frac{1}{2W}[\partial_l V\delta^i_m + \partial_m V\delta^i_l - \partial^i V\delta_{lm}] \tag{44}$$

2. The motion occurs in the subspace Γ_h of the phase space Γ; Γ_h is given by

$$\frac{1}{2}a^{ij}(q)p_ip_j + V(q) = h \tag{45}$$

$$\Gamma_h = \left\{(p, q): \frac{1}{2}a_{ij}(q)\dot{q}_i\dot{q}_j + V(q) = h\right\} \tag{46}$$

or in the subspace Q of the tangent bundle $T\mathscr{M}$, where $\mathscr{M}$ is a configuration space, Q is given by

$$Q = \left\{(q, \dot{q}): \frac{1}{2}a_{ij}\dot{q}_i\dot{q}_j + V = h\right\} \tag{47}$$

$$Q = \{(q, \dot{q}): a_{ij}(q)\dot{q}_i\dot{q}_j = 2W\}, \qquad W = |h - V| \tag{48}$$

Vector fields normalized in the sense of the metric $g_{ij} = 2Wa_{ij}$ belong to the subspace Q,

$$Q = \{(q, v_q): g(v_q, v_q) = 1\} \quad \text{for positive definite metrics } a_{ij} \tag{49a}$$

and

$$Q = \{(q, v_q): g(v_q, v_q) = \pm 1, 0\} \quad \text{for indefinite metrics } a_{ij} \tag{49b}$$

As we can see, the problem of motion in such an approach reduces itself to that of geodesics in a Riemann space $(\mathcal{M}, g)$, where $\mathcal{M}$ is the configuration space of the system, and g is a Riemann metric given by (in given coordinates)

$$g_{ij}(q) = 2W(q)a_{ij}(q) \tag{50}$$

3. A CLASS OF HAMILTONIAN SYSTEMS IN A MULTIDIMENSIONAL COSMOLOGY

When constructing multidimensional cosmological models we start with the Einstein–Maxwell action

$$S = \frac{1}{16\pi\bar{G}} \int d^D x\,(-\det \hat{g}_{\mu\nu})^{1/2}(\hat{R} - 2\hat{\Lambda}) \tag{51}$$

where $D = 1 + n + \tilde{n}$ is a total dimension of space-time, n and $\tilde{n}$ being dimensions of the physical and internal spaces, respectively; $\hat{\Lambda}$ and $\hat{G}$ are the cosmological constant and the gravitational constant, respectively.

In agreement with the Kaluza–Klein ideology, the metric of the D-dimensional space-time is assumed to be of the form

$$\hat{g}_{MN} = \begin{pmatrix} \hat{g}_{\mu\nu} & 0 \\ 0 & b^2\hat{g}_{mn} \end{pmatrix} \tag{52}$$

where $\mu, \nu = 0, 1, \ldots, n$; $m, n = 1, \ldots, \tilde{n}$; and b is only a function of time. The metric $\hat{g}_{\mu\nu}$ and the metric of physical space $g_{\mu\nu}$ are interrelated by

$$\hat{g}_{\mu\nu} = W^2 g_{\mu\nu} \tag{53}$$

where

$$W^2 = b^{\frac{2\hat{n}}{(n-1)}}$$

In the following we shall additionally assume that g_{mn} is a metric of an $\tilde{n}$-dimensional space of constant curvature, i.e.,

$$\tilde{R}_{mn} = \frac{\tilde{R}}{\tilde{n}} g_{mn} \tag{54}$$

This assumption allows us to consider the action S as resulting from the Lagrange function of the form

$$L = (-\det g_{\mu\nu})^{1/2}\left[\frac{1}{16\pi G} - \frac{1}{2}\phi_{,\lambda}, \phi^{,\lambda} - V(\phi)\right] \tag{55}$$

where

$$G = \frac{\hat{G}}{(\det g_{mn})^{1/2}}, \qquad \phi = \frac{1}{\kappa}\left[\tilde{n}\left(1 + \frac{\tilde{n}}{n-1}\right)\right]^{1/2} \ln b$$

$$\kappa = (8\pi G)^{1/2}$$

$$V(\phi) = \frac{1}{\kappa^2}\left\{\hat{\Lambda} \exp\left[-2\kappa\left(\frac{\tilde{n}}{(n-1)(\tilde{n}+n-1)}\right)^{1/2}\phi\right] - \frac{1}{2}\hat{R} \exp\left[-2\kappa\left(\frac{\tilde{n}+n-1}{\tilde{n}(n-1)}\right)^{1/2}\phi\right]\right\} \tag{56}$$

After the dimensional reduction, multidimensional world models are equivalent to those of the classical $(1+3)$-dimensional theory of gravity with minimally coupled scalar fields. We choose a coordinate system such that

$$\hat{g}_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & a^2 \bar{g}_{ij} \end{pmatrix} \tag{57}$$

where $\bar{g}_{ij}$ is the metric space of constant-curvature. After introducing new variables, we obtain

$$L = (\det \bar{g}_{ij})^{1/2} e^{n\xi}\left\{\frac{1}{\kappa^2}\left[-\frac{1}{2}n(n-1)\dot{\xi}^2 + \frac{1}{2}\bar{R}e^{-2\xi}\right] + \frac{1}{2}\dot{\phi}^2 - V(\phi)\right\} \tag{58}$$

where dot denotes differentiation with respect to t, and $\xi = \ln a$.

Hence we obtain the Hamiltonian for multidimensional world models (respectively for classical models with minimally coupled scalar fields) in the form

$$H = -\frac{1}{2}n(n-1)\dot{\xi}^2 + \frac{\kappa^2}{2}\dot{\phi}^2 - \frac{1}{2}\bar{R}e^{-2\xi} + \kappa^2 V(\phi) \tag{59}$$

together with the constraint condition $H = 0$.

The Hamiltonian flow induces a geodesic flow on pseudo-Riemannian space with the metric

$$ds^2 = 2W(d\bar{\phi}^2 - d\bar{\xi}^2) \tag{60}$$

where

$$\bar{\xi} = [2n(n-1)]^{1/2}\xi, \qquad \bar{\phi} = \kappa\phi$$

and

$$2W = \kappa^2 V\left(\frac{\bar{\phi}}{\kappa}\right) - \frac{1}{2}\bar{R}\exp\left[-\left(\frac{2}{n(n-1)}\right)^{1/2}\bar{\xi}\right] \tag{61}$$

In the next section we shall analyze the global properties of such a space.

4. KILLING TENSORS AND FIRST INTEGRALS OF MOTION

As is well known, the geodesic equation on an arbitrary manifold $\mathscr{M}$ with Killing vector (ξ_i) has linear first integrals $\xi_i u^i = \text{const}$, where $u^i = dx^i/ds$ is the unit tangent vector to a geodesic. In such a case, the field Killing vector satisfies the Killing equation

$$\xi_{i;j} + \xi_{j;i} = 0 \tag{62}$$

If there exist first integrals of second or higher orders,

$$U_{i_1 \ldots i_n} u^{i_1} \ldots u^{i_n} = \text{const}$$

then the totally symmetric field tensor $U_{i_1 \ldots i_n}$ satisfies the Killing tensor equation

$$U_{(i_1 \ldots i_n; i_{n+1})} = 0 \tag{63}$$

where parentheses indicate total symmetrization (Dolan *et al.*, 1989). Therefore, if we know Killing vectors (KVs for short) and Killing tensors (KTs) of the reduced pseudo-Riemannian spaces, we are able to construct effectively algebraic first integrals of Hamiltonian systems. With the help of KVs we obtain first integrals, linear in momenta, whereas KTs give us first integrals of higher orders in momenta.

Sommers (1973) and Geroch (1970) have shown that one can, in the simplest and most natural way, present properties of KVs and KTs as certain classes of homogeneous polynomials on the phase space, i.e., on the cotangent bundle $T^*(M_n)$ of a manifold M_n. For any two invariants $f, g \in T^*(M_n)$ a new invariant can be obtained in the form of the Poisson bracket

$$(f, g) = \frac{\partial f}{\partial x^s}\frac{\partial g}{\partial p_s} - \frac{\partial f}{\partial p_s}\frac{\partial g}{\partial x^s} = -(g, f) \tag{64}$$

[$T^*(M_n)$ is a $2n$-dimensional phase space with local coordinates (x^i, p_j)]. Another invariant can be obtained from the Jacobi identity

$$(f, (g, h)) + (g, (h, f)) + (h, (f, g)) = 0 \tag{65}$$

Invariants which are linear polynomials in momenta p_i remain in a one-to-one correspondence with the totally symmetric contravariant tensor field on M_n. Therefore, the field vector $\xi = \xi^i(\partial/\partial x^i)$ can be identified with linear polynomials $\xi^i p_i$, and the totally symmetric tensor field with polynomials

$$U = U^{i_1 \cdots i_n} p_{i_1 \ldots} p_{i_n}, \qquad n \geq 2$$

Killing vectors on $T^*(M_n)$ assume the form

$$(X, E) = 0 \tag{66}$$

where the metric tensor on M_n has been replaced by the basic invariant $E = g^{ij}p_i p_j$, i.e., by the kinetic energy of the system (Dolan, 1984).

If KVs $X_1, \ldots, X_r$ are linearly independent and form a closed set, one obtains a real Lie algebra γ of dimension r [the maximal dimension being $\frac{1}{2}n(n+1)$, $n = \dim M_n$]. The Jacobi identity has the form

$$((X_b, X_c), E) + (X_b, (X_c, E)) + (X_c, (X_b, E)) = 0 \tag{67}$$

where

$$(X_b, X_c) = \left(\xi^s_{(c)} \frac{\partial \xi^i_{(b)}}{\partial x^s} - \xi^s_{(b)} \frac{\partial \xi^i_{(c)}}{\partial x^s}\right) p_i = -[\xi_b, \xi_c] p_i \tag{68}$$

and $[\xi_b, \xi_c]$ is a Lie bracket of KVs $\xi_b = \xi^i_{(b)}\, \partial/\partial x^i$ and $\xi_c = \xi^i_{(c)}\, \partial/\partial x^i$ on M_n. If, therefore, (X_b, X_c) is linear in momenta, then p_i must be a Killing vector, and it must be a linear combination of X_a, i.e.,

$$(X_b, X_c) = -C^a_{bc} X_a; \qquad a, b, c = 1, \ldots, r \tag{69}$$

where $C^a_{bc} = -C^a_{cb}$ are constants.

KTs are defined as arbitrary solutions of the equation $(U, E) = 0$. The Schouten–Nijenhuis bracket of KT fields plays here an analogous role to that of the Lie bracket in constructing Lie algebras of KVs. In consequence, the Lie algebra of KVs extends to the graded algebra of KVs and KTs (Xanthopoulos, 1984). The same author has demonstrated that the number of irreducible KTs is likely to be very small, and that the number of quadratic KTs tends to zero as the valence u increases.

Now we shall illustrate the significance of KVs and KTs by using them to demonstrate the integrability of Liouville dynamical systems. Kinetic energy and potential energy for these systems can be expressed in the following form:

$$T = \frac{1}{2}[u_1(q_1) + \cdots + u_n(q_n)][v_1(q_1)\dot{q}_1^2 + \cdots + v_n(q_n)\dot{q}_n^2] \tag{70}$$

$$V = \frac{w_1(q_1) + \cdots + w_n(q_n)}{u_1(q_1) + \cdots + u_n(q_n)} \tag{71}$$

where $H = T + V \doteq h = \text{const}$ is the energy integral. As demonstrated in the preceding section, the problem can be reduced to the analysis of the Riemannian space with the metric

$$ds^2 = (h - V)[v_1(q_1) + \cdots + v_n(q_n)][v_1(q_1)\, dq_1^2 + \cdots + v_n(q_n)\, dq_n^2] \tag{72}$$

After introducing new variables

$$\int [v_r(q_r)]^{1/2}\, dq_r = \bar{q}_r, \qquad hv_i(q_i) - w_i(q_i) = \bar{w}_i \tag{73}$$

we must look for KVs and KTs for the following metric:

$$ds^2 = [\bar{w}_1(\bar{q}_1) + \cdots + \bar{w}_n(\bar{q}_n)][d\bar{q}_1^2 + \cdots + d\bar{q}_n^2] \tag{74}$$

Spaces with this metric are higher-dimensional generalizations of the Liouville space. To show the integrability of the Liouville dynamical system, it is enough to integrate the geodesic equation, which gives

$$\begin{aligned} &[hu_1(q_1) - w_1(q_1) + \gamma_1]^{-1/2} dq_i \pm [hu_2(q_2) - w_2(q_2) + \gamma_2]^{-1/2} dq_2 = 0 \\ &[hu_{n-1}(q_{n-1}) - w_{n-1}(q_{n-1}) + \gamma_{n-1}]^{-1/2} dq_{n-1} \\ &\qquad \pm [hu_n(q_n) - w_n(q_n) + \gamma_n]^{-1/2} dq_n = 0 \end{aligned} \tag{75}$$

where $\gamma_1 + \cdots + \gamma_n = 0$.

In this way we obtain $(n/2)$ relationships

$$C_i \frac{dq_i}{ds} \pm C_j \frac{dq_j}{ds} = 0 \tag{76}$$

where C_i are functions of q_i. Since the dimension of the Liouville system is here irrelevant, in the following we shall be interested in Liouville surfaces with the metric

$$ds^2 = [V(v) + U(u)](dv^2 + du^2) \tag{77}$$

In this metric geodesics assume the form

$$\frac{dv}{[V(v) + a]^{1/2}} \pm \frac{du}{[U(u) - a]^{1/2}} = 0 \tag{78}$$

These solutions are, of course, special instances of solutions corresponding to arbitrary dimensions of the Liouville systems. It can be shown that any surface locally isometric to a surface with a rotational symmetry is a Liouville surface (Mishchenko *et al.*, 1985).

One can also prove that for a Liouville system one has

$$\begin{aligned} &\frac{\frac{1}{2}[u_1(q_1) + \cdots + u_n(q_n)]^2 (dq_1/dt)^2}{hu_1(q_1) - w_1(q_1) + \gamma_1} \\ &= \cdots = \frac{\frac{1}{2}[u_1(q_1) + \cdots + u_n(q_n)]^2 (dq_n/dt)^2}{hu_n(q_n) - w_n(q_n) + \gamma_n} \end{aligned} \tag{79}$$

In the case of a Liouville surface this amounts to the existence of the first integral quadratic in momenta in the form

$$\bar{U} = \frac{\frac{1}{2}[U(u) + V(v)]^2}{V(v) + a} p_1^2 - \frac{\frac{1}{2}[U(u) + V(v)]^2}{U(u) - a} p_2^2 \tag{80}$$

As is easily checked, the Poisson bracket $(\bar{U}, E)$ vanishes,

$$(\bar{U}, E) = \frac{\partial \bar{U}}{\partial u}\frac{\partial E}{\partial p_1} + \frac{\partial \bar{U}}{\partial v}\frac{\partial E}{\partial p_2} - \frac{\partial \bar{U}}{\partial p_1}\frac{\partial E}{\partial u} - \frac{\partial \bar{U}}{\partial p_2}\frac{\partial E}{\partial v} \tag{81}$$

$$E = (U + V)(p_1^2 + p_2^2) \tag{82}$$

The integrability of geodesic equations (in a closed form) is a consequence of the fact that $(\bar{U}, E) = 0$. The Killing equations for KTs U assume the form

$$(U, E) = -[U, E]^{i_1 \cdots i_{k+1}} p_{i_1} \cdots p_{i_{k+1}} - \left\{ u U^{s(i_1 \cdots i_{k-1}} \frac{\partial g^{i_k i_{k+1})}}{\partial x^s} - 2g^{s(i_1} \frac{\partial U^{i_2 \cdots i_{k+1})}}{\partial x^s} \right\} p_{i_1} \cdots p_{i_{k+1}} \tag{83}$$

where $[\cdot, \cdot]$ denotes the Schouten–Nijenhuis bracket. If U and V are arbitrary KTs, then their Schouten–Nijenhuis bracket is also a KT. For totally symmetric contravariant tensor fields of valence $u + w - 1$, their Schouten–Nijenhuis bracket is defined in the following way (Sommers, 1973; Dolan, 1984):

$$[V, W]^{i_1 \cdots i_{v+w-1}} = v V^{s(i_1 \cdots i_{v-1}} \frac{\partial W^{i_v \cdots i_{v+w-1})}}{\partial x^s} - w W^{s(i_1 \cdots i_{w-1}} \frac{\partial V^{i_w \cdots i_{v+w-1})}}{\partial x^s} \tag{84}$$

In order to represent (U, V) as a constant linear combination of KTs of valence $w + v - 1$, one should distinguish reducible and irreducible KTs of any valence. Only irreducible KTs are regarded as having a physical meaning (an arbitrary totally symmetric tensor need not be irreducible). Let the number of irreducible KTs of valence u be r_u, and let us assume that the most general KT of valence u can be constructed as a constant linear combination of N_u "basic" KTs. In such a case N_u is a function of $r_u, r_{u-1}, \ldots, r_{u-2} = r_2, r_1 = r$, where r_2 includes the metric invariant E and r includes TVs.

In our case $r_1 = 0$, therefore $r_2 = N_2 = 2$. Hence γ does not exist, but there exists an extension of γ to the Γ.

Now we shall show that the existence of γ is connected with the existence of first integrals, linear in momenta, for a Hamiltonian dynamical system (with certain limitations upon their kinetic and potential energy). To show this connection it is enough to make use of the Levy (1878) theorem. Let us consider systems, the Lagrangian of which consists of the kinetic energy $T(\dot{q}_1, \ldots, \dot{q}_n, q_1, \ldots, q_n)$, which is a quadratic function in velocities, and of the potential energy $V(q_1, \ldots, q_n)$, which is independent of velocities. In order to admit a first integral linear in velocities (which is equivalent to the existence of Killing fields on the reduced manifold), the system must have a cyclic coordinate or it should be possible to transform it (with the help of pointwise transformation) into a system having such a coordinate (Whittaker, 1952). This result is known as the *Levy theorem*; it implies that if

$$T = \frac{1}{2} a_{ik}\dot{q}_i\dot{q}_k; \qquad V = V(q_1, \dots, q_n), \qquad a_{ik} = a_{ik}(q_1, \dots, q_n) \tag{85}$$

then the system admits the first integral

$$C_1\dot{q}_1 + \cdots + C_n\dot{q}_n + C = \text{const} \tag{86}$$

where $C_1, \dots, C_n, C$ are only functions of $q_1, \dots, q_n$. In such a case the first integral linear in momenta exists *if and only if* one can transform the space with the metric

$$ds^2 = a_{ik}\, dq^i\, dq^k \tag{87}$$

in such a way that one of the variables vanishes from the coefficients (Cerruti, 1907).

Moreover, if the motion of the system with the potential V and kinetic energy form $T = \frac{1}{2}\delta^{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta$ leads to geodesics in a Liouville space having the metric

$$ds^2 = [u_1(q_1) + \cdots + u_n(q_n)](dq_1^2 + \cdots + dq_n^2) \tag{88}$$

then the problem is equivalent to that of motion of a free particle with the kinetic energy of form $T = \frac{1}{2} a_{ik}\dot{q}_i\dot{q}_k$, where $a_{ik} = u\delta_{ik}$, $u = \sum_{i=1}^n u_i(q_i)$. In turn, from the Levy theorem it follows that in such a case, vanishing of one of the variables in the coefficients A_{ik} is a necessary and sufficient condition for the existence of first integrals linear in momenta. In the case of the Liouville surface this means that

$$ds^2 = U(u)(du^2 + dv^2) \tag{89}$$

or

$$ds^2 = V(v)(du^2 + dv^2) \tag{90}$$

modulo coordinate transformation.

Since any surface with a rotational symmetry

$$\mathbf{r} = \{\varphi(\vartheta)\cos(u), \varphi(\vartheta)\sin(u), \psi(\vartheta)\} \tag{91}$$

is equipped with the metric

$$ds^2 = \varphi^2(\vartheta)\, du^2 + \{\varphi'^2(\vartheta) + \psi'^2(\vartheta)\}\, d\vartheta^2 \tag{92}$$

the Liouville space is a surface with a rotation symmetry. To show this, it is enough to make the transformation

$$d\bar{\vartheta} = \frac{\varphi'^2(\vartheta) + \psi'^2(\vartheta)}{\varphi^2(\vartheta)} d\vartheta \tag{93}$$

Let us now consider a surface with a rotation symmetry which is embedded in the Euclidean space with coordinates (x, y, z). Let the surface be generated by a rotation, by angle Θ, of a function $y = f(x)$ around the x axis, i.e.,

$$(x, \Theta) \to (x, f(x) \cos \Theta, f(x) \sin \Theta) \tag{94}$$

The basis vectors $E_1 = \partial/\partial x$, $E_2 = \partial/\partial\Theta$ on this surface are

$$\begin{aligned} E_1(x, \Theta) &= (1, f'(x) \cos \Theta, f'(x) \sin \Theta) \\ E_2(x, \Theta) &= (0, -f'(x) \sin \Theta, f'(x) \cos \Theta) \end{aligned} \tag{95}$$

and consequently the components of the metric assume the form

$$g_{11}(x, \Theta) = 1 + [f'(x)]^2, \qquad g_{12} = g_{21} = 0, \qquad g_{22} = f^2(x) \tag{96}$$

KVs are solutions of the Killing equations

$$\mathscr{L}_\xi g = 0 \tag{97}$$

where $\mathscr{L}_\xi$ is a Lie derivative in a direction ξ (in our case $\boldsymbol{\xi} = E_2$).

Let us consider a tangent vector to the geodesic $\mathbf{u} = u^1 E_1 + u^2 E_2$. Since along the geodesic $\boldsymbol{\xi} \cdot \mathbf{u} = \text{const} \Rightarrow u^2 = \text{const}$, the second component of the tangent vector is a constant of motion, i.e.,

$$u^2 = \frac{\partial\Theta}{\partial s} = \text{const}$$

In other words, Θ is a cyclic variable. The metric of this surface is the one of the surface with a rotation symmetry

$$\begin{aligned} ds^2 &= \{1 + [f'(x)]^2\}(dx)^2 + f^2(x)(d\Theta)^2 \\ &= f^2(x)\left[(d\Theta)^2 + \frac{1 + [f'(x)]^2}{f^2(x)} (dx)^2\right] \end{aligned} \tag{98}$$

By transformation to a new variable

$$d\bar{x} = \frac{1 + [f'(x)]^2}{f^2(x)} dx$$

the above metric changes into a particular form of the metric of the Liouville surface.

5. SOME PROPERTIES OF THE ENSEMBLE OF MULTIDIMENSIONAL WORLD MODELS

The method elaborated in the previous sections turns out to be a suitable tool to investigate some global properties of the space of solutions of Einstein's equations. In such an approach the problem reduces to the analysis of geodesics on a Riemannian space with the metric

$$ds^2 = 2W(d\phi^2 - d\xi^2) \tag{99}$$

where

$$2W = \left| V(\phi) - \frac{1}{2}\bar{R}\exp\{-[2/n(n-1)]^{1/2}\xi\} \right| \tag{100}$$

and

$$V(\phi) = \hat{\Lambda}\exp\left[-2\left(\frac{\tilde{n}}{(n-1)(\tilde{n}+n-1)}\right)^{1/2}\phi\right] - \frac{1}{2}\hat{R}\exp\left[-2\left(\frac{\tilde{n}+n-1}{\tilde{n}(n-1)}\right)^{1/2}\phi\right] \tag{101}$$

(the natural system of units is chosen in which $\kappa = 1$).

The above metric is a special case of the metric of the Liouville surface (which is conformally flat)

$$ds^2 = [A(u) + B(w)](du^2 - dw^2) \tag{102}$$

where, in the conformal factor, dependences are separated of both variables, i.e., of the field variable $\phi = u$ and of the dynamical variable $\xi \propto \ln R$, where R is a scale factor of the physical space.

Let us notice that (as was shown above) the considered class of the systems is equivalent to the classical cosmology with the scalar field ϕ and a potential $V(\phi)$.

The geodesic equation, after changing to a new parameter $t = t(s)$ [or $s = s(t)$], has the form

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}[x(s)]\frac{dx^j}{ds}\frac{dx^k}{ds} = f(s)\frac{dx^i}{ds} \tag{103}$$

where

$$f(s) = -\left(\frac{dt}{ds}\right)^2\frac{ds^2}{dt^2} \tag{104}$$

If we put $x^1 = v$, $x^2 = w$, $s = u$, then

$$\frac{d^2x^1}{ds^2} = 0; \qquad \frac{d^2x^2}{ds^2} = \frac{d^2w}{ds^2}; \qquad \frac{dx^2}{ds} = \frac{dw}{du} \tag{105}$$

and

$$\frac{d^2w}{du^2}+\Gamma^2_{11}[u, w(u)]+2\Gamma^2_{12}[u, w(u)]\frac{dw}{du}+\Gamma^2_{22}[u, w(u)]\left(\frac{dw}{ds}\right)^2=f(u)\frac{dw}{ds} \quad (106)$$

Therefore, the geodesic equation assumes the form

$$\frac{d^2w}{du^2}+\Gamma^2_{11}[u, w(u)]+\{2\Gamma^2_{12}-\Gamma^1_{11}\}\frac{dw}{du}+\{\Gamma^2_{22}-2\Gamma^1_{12}\}\left(\frac{dw}{du}\right)^2-\Gamma^1_{22}\left(\frac{dw}{du}\right)^3=0 \quad (107)$$

where

$$\Gamma^1_{11}=\Gamma^1_{22}=\Gamma^2_{12}=\frac{1}{2[A+B]}\frac{dA}{du}$$

$$\Gamma^1_{12}=\Gamma^2_{11}=\Gamma^2_{22}=\frac{1}{2[A+B]}\frac{dB}{dw}$$

After standard manipulations one obtains

$$[A+B]\frac{d}{du}\left[\left(\frac{dw}{du}\right)^2\right]=-\frac{dB}{du}-\frac{dA}{du}\left(\frac{dw}{du}\right)^2+\frac{dB}{du}\left(\frac{dw}{du}\right)^2+\frac{dA}{du}\left(\frac{dw}{du}\right)^4 \quad (108)$$

By multiplying it by $du\ (du)^4$, one has

$$[A+B]\,du^2\,d[(dw)^2]=-dB\,du^4-dA\,du^2\,dw^2+dB\,du^2\,dw^2+dA\,dw^4$$
$$-[du^2-dw^2][dB\,(du)^2+dA\,(dw)^2] \quad (109)$$

Let $P\equiv B\,du^2+A\,dw^2$, $Q=dw^2-du^2$. In such a case one can demonstrate that $P\,dQ=Q\,dP$, i.e.,

$$d\left(\frac{P}{Q}\right)=d\left\{\frac{B\,du^2+A\,dw^2}{du^2-dw^2}\right\}=0 \quad (110)$$

After elementary transformations we obtain the following solution of the deviation equation:

$$\frac{du^2}{-[A(u)+a]}=\pm\frac{dw^2}{[B(w)-a]} \quad (111)$$

or

$$\frac{du}{\{-[A(u)+a]\}^{1/2}}\pm\frac{dw}{\{[B(w)-a]\}^{1/2}}=0 \quad (112)$$

where $A(v)+a<0$ and $B(w)-a>0$. In principle, this result could be guessed from the classical Liouville metric by substituting $dw^2\to -dw^2$. The obtained integral demonstrates the full integrability of the system. We meet here a case analogous to that of classical Liouville systems, for which integrability is a consequence of the existence of KT algebra of the second rank. Of course, $u = \phi$, $w = \xi$, and

$$A(u) = V(\phi), \qquad B(w) = B(\xi)$$

In general, $w = w(\phi, \xi)$ is a three-parameter function depending on the cosmological constant $\hat{\Lambda}$, on the constant curvature $\tilde{R}$ of the internal space, and on the constant curvature $\bar{R}$ of the physical space. The function $W(\phi, \xi)$ is constructed out of functions of the type

$$\alpha e^{\beta\phi}; \qquad \bar{\alpha} e^{\bar{\beta}\xi}$$

where $\alpha > 0$ and $\bar{\alpha} > 0$ in such a way that if any two arbitrary parameters vanish, the Ricci scalar of the corresponding space (to which the ensemble is reduced) vanishes, too. To see this, it is enough to notice that the transformation of the variables

$$\begin{aligned} \bar{\phi} &= \frac{\sqrt{\alpha}}{\beta} e^{\beta\phi} \cosh(\beta\phi) \\ \bar{\xi} &= \frac{\sqrt{\alpha}}{\beta} e^{\beta\phi} \sinh(\beta\phi) \end{aligned} \tag{113}$$

changes the metric (102) into the Minkowski metric

$$ds^2 = d\bar{\phi}^2 - d\bar{\xi}^2 \tag{114}$$

If the internal space is Ricci flat, $\tilde{R} = 0$ (i.e., if it is a torus), then the signature of the Ricci scalar is equal to $\text{sign}(\bar{R}\hat{\Lambda})$. In the case of $\hat{\Lambda} = 0$ and $n = 0$ (the physical space is three-dimensional), the sign of the Ricci scalar is independent of the field ϕ and is equal to $\text{sign}(R) = -\text{sign}(\tilde{R}\bar{R})$. Analogously, if $\bar{R} = 0$ (the physical space is Ricci flat), then $\text{sign}(R) = \text{sign}(\tilde{R}\hat{\Lambda})$.

If in the metric (102) one of the functions $A(u)$, $B(w)$ vanishes, i.e., if $\hat{\Lambda} = \hat{R} = 0$ or $\bar{R} = 0$, then the surface on which this metric can be realized is a surface of a rotational symmetry which originates from the rotation of the function $x = f(\bar{\phi})$, around $\bar{\phi}$ axis, in a flat pseudo-Riemannian space with the metric

$$ds^2 = d\bar{\phi}^2 - dx^2 - dy^2 \tag{115}$$

Let us consider a surface with a rotational symmetry

$$(\bar{\phi}, \xi) \to (\bar{\phi}, f(\bar{\phi}) \cos \xi, f(\bar{\phi}) \sin \xi) \tag{116}$$

We have

$$ds^2 = \{1 - [f'(\bar{\phi})]^2\} \, d\bar{\phi}^2 - f^2(\bar{\phi}) \, d\xi^2 \tag{117}$$

where prime denotes differentiation with respect to $\bar{\phi}$. After changing to a new variable $\tilde{\phi}$ defined through

$$d\tilde{\phi} = \left\{ \frac{[f'(\bar{\phi})]^2}{f^2(\bar{\phi})} \right\}^{1/2} d\bar{\phi} \tag{118}$$

we obtain

$$ds^2 = F^2(\tilde{\phi})[d\tilde{\phi}^2 - d\xi^2]^2 \tag{119}$$

Hamiltonian systems which can be reduced to a space with the metric (114) admit Killing fields generating a first integral linear in momenta. Levine (1936, 1939) formulated conditions for the conformal factor S: $g_{ij} = S^2 \eta_{ij}$ [i.e., $S = (2W)^{1/2}$] for which KVs exist. Having the above in mind, we can fully describe the class of Hamiltonian dynamical systems of the gravity theory for which first integrals quadratic in momenta exist (the characteristic feature of this class is that their quadratic form of kinetic energy is indefinite).

6. CONCLUSIONS

We have investigated the problem of the geometrization of the dynamics of homogeneous multidimensional cosmological models by reducing the class of these models to Hamiltonian geodesic flows. As is well known, the local instability of geodesic flows on compact and zero-energy surfaces (in general constant-energy surfaces) leads to ergodicity. In our case, surfaces of constant energy are not compact. It turns out that the local stability of a geodesic flow is determined by the sign of the sectional curvature of the reduced space (of the Gaussian curvature in our case). This follows from the geodesic deviation equation for the perpendicular component of the geodesic separation vector (Szydłowski and Łapeta, 1991). If the Ricci scalar is negative, nearby geodesics diverge, in average exhibiting the local instability provided $\tilde{R}\bar{R} > 0$ if $\hat{\Lambda} = 0$. Let us suppose that nearby geodesic converge in average; in such a case one can say that if a certain geodesic, corresponding to a given trajectory of the Hamiltonian system, exhibits a certain property, then a nearby geodesic will exhibit the same property. For instance, if a multidimensional world model admits inflation as a dynamical effect of extra dimensions (Szydłowski *et al.*, 1993), the inflation is typical if $\tilde{R}\bar{R} < 0$, i.e., if the physical space is of a negative curvature. For such a class of world models, one has an interesting property of passing through an infinite series of inflationary epochs. In general, based on the geodesic deviation equation one can discuss Lyapunov stability of certain properties of world models. This idea has been developed in the present work. Our main results are the following:

1. The dynamics of cosmological models has been translated into purely geometric language.
2. It turns out that Killing vectors and Killing tensors are important in determining additional first integrals (besides the energy integral).
3. The proof of the integrability has been offered of the Hamiltonian dynamical systems of a multidimensional cosmology by integrating the corresponding geodesic deviation equation.
4. Classical Liouville systems have been generalized to the case of generally relativistic systems, the energy form of which is indeterminate. It turns out that the integrability of such systems follows from the existence of the Killing tensor algebra leading to integrals of second order in momenta.
5. The problem of the existence of first integrals linear in momenta has been formulated in terms of Killing vectors of the isometry group.

We believe that our main result can be expressed as a proposal that instead of investigating Hamiltonian flows one should study the behavior of geodesics in a certain model of Hamiltonian systems obtained with the help of the *Maupertuis–Jacobi–Hamilton* principle. In this model the integrability of a system manifests itself in the symmetries of the model, and the nonintegrability of a system in the lack of symmetries.

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